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Absence of nontrivial solutions for a class of partial differential equations and systems in unbounded domains

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Abstract

In this paper, we are interested on the study of the nonexistence of nontrivial solutions for a class of partial differential equations, in unbounded domains. This leads us to extend these results to m-equations systems. The method used is based on energy type identities.

Keywords: Differential equations, trivial solution, energy type identities.

1 Introduction

The study of the nonexistence of nontrivial solutions of partial differential equations and systems is the subject of several works of many authors, by using various methods to obtain the necessary and sufficient conditions, so the studied problems admit only the null solutions. The works of Esteban & Lions [2], Pohozaev [6] and Van Der Vorst [7], contains results concerning the semilinear elliptic equations and systems. A semilar result can be found in [4], where studied equations one of the form

$$\begin{cases} \lambda\frac{\partial^2 u}{\partial t^2} - \frac{\partial}{\partial x}\left(p\left(x,y\right)\frac{\partial u}{\partial x}\right) - \frac{\partial}{\partial y}\left(q\left(x,y\right)\frac{\partial u}{\partial y}\right) + f\left(x,y,u\right) = 0 \text{ in } \mathbb{R}\times\omega, \\ u + \varepsilon\,\frac{\partial u}{\partial n} = 0 \text{ on } \mathbb{R}\times\partial\omega, \end{cases}$$

considered in $H^2(\mathbb{R}\times\omega)\cap L^\infty(\mathbb{R}\times\omega)$, where $\omega =]a_1, b_1[\times]a_2, b_2[$ and this equation does not admit nontrivial solutions if the following conditions holds

$$f(0) = 0,\ 2F(u) - uf(u) \le 0.$$

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In this work similar results for a class of the partial differential equations and systems were also obtained.

Let us consider the following problem in $H^2(\mathbb{R}\times\Omega)\cap L^\infty(\mathbb{R}\times\Omega)$, Ω a bounded domain of $\mathbb{R}^n$, for a function $\lambda:\mathbb{R}\to\mathbb{R}$, not changing sign and $p:\mathbb{R}\times\Omega\to\mathbb{R}$ also had not changing sign.

$$\begin{cases} -\frac{\partial}{\partial t}\left(\lambda\left(t\right)\frac{\partial u}{\partial t}\right)-\sum\limits_{i=1}^{n}\frac{\partial}{\partial x_i}\left(p\left(t,x\right)\frac{\partial u}{\partial x_i}\right)+f\left(x,u\right)=0 \text{ in } \mathbb{R}\times\Omega, \\ u+\varepsilon\frac{\partial u}{\partial \nu}=0 \text{ on } \mathbb{R}\times\partial\Omega. \end{cases} \tag{1.1}$$

We use the notations

$$\begin{aligned} &H=L^2\left(\Omega\right), \\ &\left\|u\left(t,x\right)\right\|=\left(\int\limits_{\Omega}\left|u(t,x)\right|^2 dx\right)^{\frac{1}{2}}, \text{ the norm of } u \text{ in } H, \\ &\left\|\nabla u\left(t,x\right)\right\|^2=\int\limits_{\Omega}\sum\limits_{i=1}^{n}\left|\frac{\partial u}{\partial x_i}\right|^2 dx, \\ &F(x,u)=\int\limits_0^u f(x,\sigma)d\sigma,\ \forall\ x\in\Omega,\ u\in\mathbb{R}. \end{aligned}$$

Let L be the operator defined by

$$Lu\left(t,x\right)=-\sum\limits_{i=1}^{n}\frac{\partial}{\partial x_i}\left(p\left(t,x\right)\frac{\partial u}{\partial x_i}\right),\left(t,x\right)\in\mathbb{R}\times\Omega,$$

and $f:\overline{\Omega}\times\mathbb{R}\to\mathbb{R}$ a real continuous function, locally Lipschitz in u, such that

$$f(x,0)=0,\ \forall\ x\in\overline{\Omega}.$$

We assume that

$$u\in H^2(\mathbb{R};H)\cap L^\infty(\mathbb{R};L^\infty(\Omega)),$$

satisfies the equation

$$-\frac{\partial}{\partial t}\left(\lambda\left(t\right)\frac{\partial u}{\partial t}\left(t\right)\right)+Lu\left(t,x\right)+f(x,u)=0\ ,\left(t,x\right)\in\mathbb{R}\times\Omega, \tag{1.2}$$

under the boundary conditions

$$(u+\varepsilon\tfrac{\partial u}{\partial n})\left(t,\sigma\right)=0,\left(t,\sigma\right)\in\mathbb{R}\times\partial\Omega,\ \text{Robin condition}, \tag{1.3}$$
$$u(t,\sigma)=0,\left(t,\sigma\right)\in\mathbb{R}\times\partial\Omega,\ \text{Dirichlet condition}, \tag{1.4}$$
$$\frac{\partial u(t,\sigma)}{\partial n}=0,\left(t,\sigma\right)\in\mathbb{R}\times\partial\Omega,\ \text{Neumann condition}. \tag{1.5}$$

We extend the above result of (1.1) to the system of m-equations of the form

$$\begin{cases} -\frac{\partial}{\partial t}\left(\lambda\left(t\right)\frac{\partial u_k}{\partial t}\right)-\sum\limits_{i=1}^{n}\frac{\partial}{\partial x_i}\left(p_k\left(t,x\right)\frac{\partial u_k}{\partial x_i}\right)+f_k\left(x,u_1,...,u_m\right)=0 \text{ in } \mathbb{R}\times\Omega, \\ u_k+\varepsilon\frac{\partial u_k}{\partial \nu}=0 \text{ on } \mathbb{R}\times\partial\Omega. \end{cases} \tag{1.6}$$

$1 \leq k \leq m$, where $f_k : \overline{\Omega} \times \mathbb{R}^m \rightarrow \mathbb{R}$,are real continuous functions, locally Lipschitz in u_i, verifing

$$\begin{array}{l} f_k(x, u_1, ..., 0, ..., u_m) = 0, \ \forall \ x \in \overline{\Omega}, \\ \exists F_m : \overline{\Omega} \times \mathbb{R}^m \rightarrow \mathbb{R} \text{ such that } \frac{\partial F_m}{\partial s_j} = f_j\left(x, s_1, ..., s_m\right), 1 \leq j \leq m, \end{array}$$

Let L_k be the operators defined by

$$L_k u\left(t,x\right) = -\sum_{i=1}^{n} \frac{\partial}{\partial x_i} \left(p_k\left(t,x\right) \frac{\partial u}{\partial x_i}\right), (t,x) \in \mathbb{R} \times \Omega,$$

we assume that

$$u_k \in H^2(\mathbb{R}; H) \cap L^\infty(\mathbb{R}; L^\infty(\Omega)),$$

are solutions of the system

$$-\frac{\partial}{\partial t}\left(\lambda\left(t\right)\frac{\partial u_k}{\partial t}\left(t\right)\right) + \sum_{k=1}^{m} L_k u_k\left(t,x\right) + f(x,u_1,...,u_m) = 0\ ,(t,x) \in \mathbb{R}\times\Omega, \tag{1.7}$$

$1 \leq k \leq m$, with boundary conditions

$$\begin{array}{ll} (u_k + \varepsilon\frac{\partial u_k}{\partial n})\left(t,\sigma\right) = 0, (t,\sigma) \in \mathbb{R}\times\partial\Omega, \text{ Robin condition,} & (1.8) \\ u_k(t,\sigma) = 0, (t,\sigma) \in \mathbb{R}\times\partial\Omega, \text{ Dirichlet condition,} & (1.9) \\ \frac{\partial u_k(t,\sigma)}{\partial n} = 0, (t,\sigma) \in \mathbb{R}\times\partial\Omega, \text{ Neumann condition.} & (1.10) \end{array}$$

According to the sign of λ, this type of problems comprises equations of both hyperbolic or elliptic type.

Our proof is based on energy type identities established in section 2, which make it possible to obtain the main nonexistence result in section 3. In section 4 we apply the results to some examples.

2 Identities of energy type

In this section, we give essential lemmas for showing the main result of this paper.

Lemma 1 *Let λ and p satisfy*

$$\begin{array}{l} \lambda'\left(t\right) \leq 0 \ \ (resp \ \geq 0)\,, \forall t \in \mathbb{R}, \\ \frac{\partial p}{\partial t}(t,x) \leq 0 \ \ (resp \ \geq 0)\,, \forall\left(t,x\right) \in \mathbb{R}\times\Omega. \end{array} \tag{2.1}$$

Then the following energ identity,

$$\begin{array}{l} -\frac{1}{2}\lambda\left(t\right)\left\|\frac{\partial u}{\partial t}\left(t,x\right)\right\|^2 + \frac{1}{2}\int\limits_{\Omega} p(t,x)\left|\nabla u\right|^2 dx \\ + \int\limits_{\Omega} F(x,u)dx + \dfrac{1}{2\varepsilon}\int\limits_{\partial\Omega} p(t,s)u^2\left(t,s\right)ds = 0. \end{array} \tag{2.2}$$

holds for any solution of the Robin problem of $(1.2) - (1.3)$.

Proof. The assumptions $f \in W^{1,\infty}_{loc}(\Omega\times\mathbb{R}), p \in L^{\infty}(\mathbb{R}\times\Omega)$ and $f(x,0) = 0, \forall x \in \Omega$, allow us to deduce the existence of two positive constants C_1 and C_2, such that

$$|p(t,x)| \le C_1, \ \ |F(x,u)| \le C_2 \left|u\left(t,x\right)\right|^2.$$

In addition, consider the functions

$$\Psi\left(t\right) = \frac{1}{2}\int_{\Omega} p(t,x)\left|\nabla u\right|^2 dx, \ \ \Phi\left(t\right) = \int_{\Omega} F(x,u)dx, t \in \mathbb{R},$$

where Φ and Ψ are of class C^1,and

$$\left|\Psi\left(t\right)\right| \le C_1 \left\|\nabla u(t,x)\right\|^2, \ \ \left|\Phi\left(t\right)\right| \le C_2 \left\|u(t,x)\right\|^2, \forall t \in \mathbb{R}.$$

Then,

$$\Phi'\left(t\right) = \int_{\Omega} f(x,u)\frac{\partial u}{\partial t}dx, \forall t \in \mathbb{R},$$

and

$$\begin{aligned}\Psi'\left(t\right) &= \int_{\Omega}\left(\sum_{i=1}^{n} p(t,x)\frac{\partial u}{\partial x_i}\frac{\partial^2 u}{\partial x_i \partial t} + \frac{1}{2}\frac{\partial p}{\partial t}(t,x)\left|\nabla u\right|^2\right)dx \\ &= -\int_{\Omega}\sum_{i=1}^{n}\frac{\partial}{\partial x_i}\left(p(t,x)\frac{\partial u}{\partial x_i}\right)\frac{\partial u}{\partial t}dx + \frac{1}{2}\int_{\Omega}\frac{\partial p}{\partial t}(t,x)\left|\nabla u\right|^2 dx + \int_{\partial\Omega} p(t,s)\frac{\partial u}{\partial \nu}\frac{\partial u}{\partial t}\left(t,s\right)ds.\end{aligned}$$

Define the function $K : \mathbb{R} \to \mathbb{R}$ by

$$K\left(t\right) = -\frac{1}{2}\lambda\left(t\right)\left\|\frac{\partial u}{\partial t}\left(t,x\right)\right\|^2 + \Psi(t) + \Phi(t).$$

The function K is absolutely continuous and differentiable in $\mathbb{R}$, and

$$\begin{aligned}K'(t) &= -\frac{1}{2}\lambda'\left(t\right)\left\|\frac{\partial u}{\partial t}\left(t,x\right)\right\|^2 - \lambda\left(t\right)\int_{\Omega}\frac{\partial u}{\partial t}\frac{\partial^2 u}{\partial t^2}dx + \Psi'(t) + \Phi'(t) \\ &= \frac{1}{2}\lambda'\left(t\right)\left\|\frac{\partial u}{\partial t}\left(t\right)\right\|^2 + \frac{1}{2}\int_{\Omega}\frac{\partial p}{\partial t}(t,x)\left|\nabla u\right|^2 dx + \int_{\partial\Omega} p(t,s)\frac{\partial u}{\partial \nu}\frac{\partial u}{\partial t}\left(t,s\right)ds \\ &+ \int_{\Omega}\left(-\frac{\partial}{\partial t}\left(\lambda\left(t\right)\frac{\partial u}{\partial t}\right) - \sum_{i=1}^{n}\frac{\partial}{\partial x_i}\left(p(t,x)\frac{\partial u}{\partial x_i}\right) + f(x,u)\right)\frac{\partial u}{\partial t}dx.\end{aligned}$$

Because u is solution of $(1.2) - (1.3)$, we deduce that

$$K'(x) = \frac{1}{2}\lambda'\left(t\right)\left\|\frac{\partial u}{\partial t}\left(t,x\right)\right\|^2 + \frac{1}{2}\int_{\Omega}\frac{\partial p}{\partial t}\left|\nabla u\right|^2 dx + \int_{\partial\Omega} p(t,x)\frac{\partial u}{\partial \nu}\frac{\partial u}{\partial t}\left(t,s\right)ds,$$

while on the boundary,

$$\int_{\partial\Omega} p(t,x)\frac{\partial u}{\partial \nu}\frac{\partial u}{\partial t}\left(t,s\right)ds = -\frac{1}{\varepsilon}\int_{\partial\Omega} p(t,s)\frac{\partial u}{\partial t}u\left(t,s\right)ds.$$

Also

$$\begin{aligned}K'(t) &= \frac{1}{2}\lambda'\left(t\right)\left\|\frac{\partial u}{\partial t}\left(t,x\right)\right\|^2 + \frac{1}{2}\int_{\Omega}\frac{\partial p}{\partial t}(t,x)\left|\nabla u\right|^2 dx - \frac{1}{\varepsilon}\int_{\partial\Omega} p(t,s)\frac{\partial u}{\partial t}u\left(t,s\right)ds \\ &= \frac{1}{2}\lambda'\left(t\right)\left\|\frac{\partial u}{\partial t}\left(t,x\right)\right\|^2 + \frac{1}{2}\int_{\Omega}\frac{\partial p}{\partial t}(t,x)\left|\nabla u\right|^2 dx \\ &- \frac{1}{2\varepsilon}\frac{\partial}{\partial t}\left(\int_{\partial\Omega} p(t,s)u^2\left(t,s\right)ds\right) + \frac{1}{2\varepsilon}\int_{\partial\Omega}\frac{\partial p}{\partial t}(t,s)u^2\left(t,s\right)ds,\end{aligned}$$

i.e

$$\frac{d}{dt}\left(K(t)+\frac{1}{2\varepsilon}\int\limits_{\partial\Omega}p(t,s)u^2(t,s)\,ds\right)=\frac{1}{2}\lambda'(t)\left\|\frac{\partial u}{\partial t}(t,x)\right\|^2$$

$$+\frac{1}{2}\int\limits_{\Omega}\frac{\partial p}{\partial t}(t,x)\left|\nabla u\right|^2dx+\frac{1}{2\varepsilon}\int\limits_{\partial\Omega}\frac{\partial p}{\partial t}(t,s)u^2(t,s)\,ds.$$

We set

$$M(t)=K(t)+\frac{1}{2\varepsilon}\int\limits_{\partial\Omega}p(t,s)u^2(t,s)\,ds.$$

Conditions (2.1) imply that

$$M'(t)\leq 0\ \ (\text{resp}\geq 0),\forall t\in\mathbb{R},$$

i.e H is monotonous. But also this function verifies

$$\lim_{|t|\to+\infty}M(t)=0,$$

because $M\in L^2(\mathbb{R})$. Hence $M(t)=0,\forall t\in\mathbb{R}$, and this gives the desired result. ■

Lemma 2 *Let* λ *and* p *verify* (2.1)*. The solution of the Dirichlet problem* $(1.2),(1.4)$ *or the Neumann problem* $(1.2),(1.5)$*, satisfies the following energ identity*

$$-\frac{1}{2}\lambda(t)\left\|\frac{\partial u}{\partial t}(t,x)\right\|^2+\frac{1}{2}\int\limits_{\Omega}p(t,x)\left|\nabla u\right|^2dx+\int\limits_{\Omega}F(x,u)dx=0. \tag{2.3}$$

Proof. For the problem $(1.2),(1.4)$, the fact that $u=0$ on the boundary implies that

$$\begin{array}{l}\int\limits_{\partial\Omega}p(t,x)\frac{\partial u}{\partial\nu}\frac{\partial u}{\partial t}(t,s)\,ds=\frac{d}{dt}\left(\int\limits_{\partial\Omega}p(t,x)\frac{\partial u}{\partial\nu}u(t,s)\,ds\right)\\ -\int\limits_{\partial\Omega}\frac{\partial p}{\partial t}\frac{\partial u}{\partial\nu}u(t,s)\,ds-\int\limits_{\partial\Omega}p(t,x)\frac{\partial^2 u}{\partial t\partial\nu}u(t,s)\,ds=0.\end{array}$$

For the problem $(1.2),(1.5)$, the fact that $\frac{\partial u}{\partial\nu}=0$ on the boundary implies that

$$\int\limits_{\partial\Omega}p(t,x)\frac{\partial u}{\partial\nu}\frac{\partial u}{\partial t}(t,s)\,ds=0.$$

The remainder of the proof is similar to that of Lemma 1. ■

Lemma 3 *Let* λ *and* p_k *satisfy*

$$\begin{array}{l}\lambda'(t)\leq 0\ \ (resp\ \geq 0),\forall t\in\mathbb{R},\\ \frac{\partial p_k}{\partial t}(t,x)\leq 0\ \ (resp\ \geq 0),1\leq k\leq m,\forall(t,x)\in\mathbb{R}\times\Omega.\end{array} \tag{2.4}$$

Then any solutions of the system $(1.7)-(1.8)$ *satisfies for all* $t\in\mathbb{R}$*, the following energetic identity*

$$\begin{array}{l}-\frac{1}{2}\sum_{k=1}^m\lambda(t)\left\|\frac{\partial u_k}{\partial t}(t,x)\right\|^2+\frac{1}{2}\sum_{k=1}^m\int\limits_{\Omega}p_k(t,x)\left|\nabla u_k\right|^2dx\\ +\int\limits_{\Omega}F_m(x,u_1,...,u_m)dx+\dfrac{1}{2\varepsilon}\sum_{k=1}^m\int\limits_{\partial\Omega}p_k(t,s)u_k^2(t,s)\,ds=0.\end{array} \tag{2.5}$$

Lemma 4 *Let λ and p_k verify* (2.5)*. Then the solutions of the systems* $(1.7)-(1.9)$ *or* $(1.7)-(1.10)$*, satisfies for all $t\in\mathbb{R}$, the following estimate*

$$-\frac{1}{2}\sum_{k=1}^{m}\lambda(t)\left\|\frac{\partial u_k}{\partial t}(t,x)\right\|^2+\frac{1}{2}\sum_{k=1}^{m}\int_\Omega p_k(t,x)\left|\nabla u_k\right|^2dx+\int_\Omega F_m(x,u_1,...,u_m)dx=0. \tag{2.6}$$

Proof. Let us define the function $K_m:\mathbb{R}\to\mathbb{R}$ by

$$K_m(t)=-\frac{1}{2}\sum_{k=1}^{m}\lambda(t)\left\|\frac{\partial u_k}{\partial t}(t,x)\right\|^2+\Psi_m(t)+\Phi_m(t),$$

where the functions Ψ_m and Φ_m are defined as follows

$$\begin{aligned}\Psi_m(t)&=\frac{1}{2}\int_\Omega\sum_{k=1}^{m}p_k(t,x)\left|\nabla u_k\right|^2dx,t\in\mathbb{R},\\ \Phi_m(t)&=\int_\Omega F_m(x,u_1,...,u_m)dx,t\in\mathbb{R},\end{aligned}$$

the rest of the proof is similar to the proofs of the preceding lemmas. ■

3 The main Result

Theorem 1 *Let us suppose that λ, F and f verify*

$$\begin{aligned}&\lambda(t)>0\ \ (resp<0),\forall t\in\mathbb{R},\\ &2F(x,u)-uf(x,u)\leq 0\ \ (resp\geq 0),\end{aligned} \tag{3.1}$$

and (2.1) holds. Then the problem $(1.2)-(1.3)$ admit only the null solution.

Proof. Let us define the function E by

$$E(t)=\left\|u(x,t)\right\|^2.$$

Multiplying equation (1.1) by u and integrating the new equation on Ω, we obtains

$$\begin{aligned}&\int_\Omega\left[-\frac{\partial}{\partial t}\left(\lambda(t)\frac{\partial u}{\partial t}\right)-\sum_{i=1}^{n}\frac{\partial}{\partial x_i}\left(p(t,x)\frac{\partial u}{\partial x_i}\right)-f(x,u)\right]udx\\ &=\int_\Omega\left[-\frac{1}{2}\left(\lambda'(t)\frac{\partial(u^2)}{\partial t}+\lambda(t)\frac{\partial^2(u^2)}{\partial t^2}\right)+\lambda(t)\left(\frac{\partial u}{\partial t}\right)^2\right.\\ &\left.+p(t,x)\left|\nabla u\right|^2+uf(x,u)\right]dx-\sum_{i=1}^{n}\int_{\partial\Omega}p(t,s)\frac{\partial u}{\partial x_i}u(t,s)\nu_i ds\\ &=-\frac{1}{2}\left(\lambda(t)E''(t)+\lambda'(t)E'(t)\right)+\lambda(t)\left\|\frac{\partial u}{\partial t}(t,x)\right\|^2+\int_\Omega p(t,x)\left|\nabla u\right|^2dx\\ &+\int_\Omega uf(x,u)dx-\int_{\partial\Omega}p(t,s)\frac{\partial u}{\partial\nu}u(t,s)ds=0.\end{aligned}$$

Using identity (2.2), we have

$$\begin{aligned}&\frac{d}{dt}\left(\lambda(t)E'(t)\right)=\lambda(t)E''(t)+\lambda'(t)E'(t)\\ &=2\lambda(t)\left\|\frac{\partial u}{\partial t}(t)\right\|^2+2\int_\Omega p(t,x)\left|\nabla u(t,x)\right|^2dx\\ &+2\int_\Omega u(t,x)f(x,u(t,x))dx+\frac{2}{\varepsilon}\int_{\partial\Omega}p(t,s)u^2(t,s)ds\\ &=4\lambda(t)\left\|\frac{\partial u}{\partial t}(t)\right\|^2-2\int_\Omega(2F(x,u)-uf(x,u))dx.\end{aligned}$$

If $\lambda(t) > 0$, the assumption (3.1) implies that

$$\frac{d}{dt}\left(\lambda(t)\, E'(t)\right) = \lambda(t)\, E''(t) + \lambda'(t)\, E'(t) \geq 0 \ , \forall t \in \mathbb{R}. \qquad (3.2)$$

We conclude that

$$E'(t) \leq 0,$$

otherwise,

$$\exists t_1 \geq 0, E'(t_1) \geq 0. \qquad (3.3)$$

Equation (3.2) implies that $\lambda(t)\, E'(t)$ is an increasing function

$$\lambda(t_1)\, E'(t_1) \leq \lambda(t)\, E'(t)\, , \forall t \geq t_1,$$

but, one has

$$\lim_{|t|\to+\infty} E'(t) = 0,$$

because $E' \in L^2(\mathbb{R})$, then

$$\lambda(t_1)\, E'(t_1) \leq 0 \text{ and } \lambda(t_1) > 0 \Rightarrow E'(t_1) \leq 0,$$

which contradicts relation (3.3). Hence,

$$E'(t) \leq 0, \forall t \in \mathbb{R}$$

i.e E is monotonous. But, this function verifies

$$\lim_{|t|\to+\infty} E(t) = 0,$$

witch implies that

$$E(t) = 0, \forall t \in \mathbb{R},$$

Thus $u = 0$ in $\mathbb{R} \times \nleq$.

If $\lambda(t) < 0$, we deduce by the same manner that $u = 0$ in $\mathbb{R} \times \nleq$. ■

Theorem 2 *Let λ, F and f verify* (3.1) *and* (2.1) *holds. Then the only solution of the problems* $(1.2) - (1.4)$ *or* $(1.2) - (1.5)$ *is the null solution.*

Proof. Identical to that of Theorem 1. ■

Theorem 3 *Let us suppose that λ, F_m and $f_k, 1 \leq k \leq m$, satisfy*

$$\begin{array}{l} \lambda(t) > 0 \ \ (resp \ < 0)\, , \forall t \in \mathbb{R}, \\ F_m(x, u_1, ..., u_m) - \sum_{k=1}^{m} u_k f_k(x, u_1, ..., u_m) \leq 0 \ \ (resp \ \geq 0)\, , \end{array} \qquad (3.4)$$

and (2.5) holds. Then the system $(1.7) - (1.8)$ admit only the null solutions.

Proof. Multiplying equation (1.6) by u_k and integrating the new equation on Ω, one obtains

$$-\frac{1}{2}\left(\lambda(t)\frac{d^2}{dt^2}\|u_k(t,x)\|^2 + \lambda'(t)\frac{d}{dt}\|u_k(t,x)\|^2\right) + \lambda(t)\left\|\frac{\partial u_k}{\partial t}(t,x)\right\|^2 + \int_\Omega p_k(t,x)\,|\nabla u_k|^2\, dx$$
$$+\int_\Omega u_k f_k(x, u_1, ..., u_m)\, dx - \int_{\partial\Omega} p_k(t,s)\,\frac{\partial u_k}{\partial \nu} u_k(t,s)\, ds = 0.$$

The sum on k from 1 to m gives

$$-\frac{1}{2}\left(\lambda(t)E_m''(t)+\lambda'(t)E_m'(t)\right)+\lambda(t)\sum_{k=1}^m\left\|\frac{\partial u_k}{\partial t}(t,x)\right\|^2+\sum_{k=1}^m\int_\Omega p_k(t,x)\left|\nabla u_k\right|^2dx$$
$$+\sum_{k=1}^m\int_\Omega u_k(t,x)f_k(x,u_1,...,u_m)dx-\sum_{k=1}^m\int_{\partial\Omega}p_k(t,s)\frac{\partial u_k}{\partial\nu}u_k(t,s)ds=0.$$

By using identity (2.6), we deduce that

$$\frac{d}{dt}\left(\lambda(t)E_m'(t)\right)=\lambda(t)E_m''(t)+\lambda'(t)E_m'(t)$$
$$=4\lambda(t)\sum_{k=1}^m\left\|\frac{\partial u_k}{\partial t}(t)\right\|^2-2\int_\Omega\left(2F_m(x,u_1,...,u_m)-\sum_{k=1}^m u_k(t,x)f_k(x,u_1,...,u_m)\right)dx.$$

Then the assumption (3.4) gives the result. ■

Theorem 4 *Let λ, p and F verify*

$$\begin{array}{l}\lambda(t)>0,p(t,x)<0 \text{ and } F(x,u)\leq 0,\forall(t,x)\in\mathbb{R}\times\Omega,\\ \text{or}\\ \lambda(t)<0,p(t,x)>0 \text{ and } F(x,u)\geq 0,\forall(t,x)\in\mathbb{R}\times\Omega,\end{array}\qquad(3.5)$$

and (2.1) *holds. Then the problems* $(1.2)-(1.3),(1.2)-(1.4)$ *and* $(1.2)-(1.5)$ *admit only the null solution.*

Proof. Assumptions (3.4) and equality (2.3) allow is

$$-\frac{1}{2}\lambda(t)\left\|\frac{\partial u}{\partial t}(t,x)\right\|^2+\frac{1}{2}\int_\Omega p(t,x)\left|\nabla u\right|^2dx+\int_\Omega F(x,u)dx=0,$$

which implies that

$$\frac{\partial u}{\partial t}(t,x)=0,\forall(t,x)\in\mathbb{R}\times\Omega,$$

i.e.

$$u(t,x)=u(x).$$

But the following condition is necessary

$$\int_{\mathbb{R}\times\Omega}\left|u(t,x)\right|^2dtdx=\int_{\mathbb{R}\times\Omega}\left|u(x)\right|^2dtdx<+\infty,$$

then $u\equiv 0$. ■

Theorem 5 *Let λ, p_k $(1\leq k\leq m)$ and f satisfy*

$$\begin{array}{l}\lambda(t)>0,p_k(t,x)<0 \text{ and } F_m(x,u_1,...,u_m)\leq 0,\forall(t,x)\in\mathbb{R}\times\Omega,\\ \text{or}\\ \lambda(t)<0,p_k(t,x)>0 \text{ and } F_m(x,u_1,...,u_m)\geq 0,\forall(t,x)\in\mathbb{R}\times\Omega.\end{array}\qquad(3.6)$$

and (2.4) *holds. Then the systems* $(1.7)-(1.8),(1.7)-(1.9)$ *and* $(1.7)-(1.10)$ *admit only the null solutions.*

Proof. Similar to that of Theorem 2. ■

Remark 1 *Note that one can apply these results in the field $\mathbb{R}^+\times\Omega$, with the condition*

$$u(0,x)=0,\forall x\in\Omega.$$

4 Applications

Example 1 *Let*

$$\theta, \theta_1, \theta_2 : \overline{\Omega} \to \mathbb{R},$$

be a nonnegative functions of class $C(\mathbb{R})$, $p, q \geq 1$ *and* $m \in \mathbb{R}$, *such that*

$$f(x,u) = mu + \theta_1(x)|u|^{p-1}u + \theta_2(x)|u|^{q-1}u.$$

Then the problem defined by

$$\begin{cases} \frac{\partial^2 u}{\partial t^2} - \sum\limits_{i=1}^{n} \frac{\partial}{\partial x_i}\left(\theta(x)\frac{\partial u}{\partial x_i}\right) + f(x,u) = 0 \text{ in } \mathbb{R} \times \Omega, \\ \left(u + \varepsilon \frac{\partial u}{\partial n}\right)(x,\sigma) = 0 \text{ on } \mathbb{R} \times \partial\Omega, \end{cases} \tag{4.1}$$

admits anly the null solution.

In this case it suffice to check that

$$\begin{array}{l} 2F(x,u) - uf(x,u) = \\ \theta_1(x)\left(\frac{2}{p+1} - 1\right)|u|^{p+1} + \theta_2(x)\left(\frac{2}{q+1} - 1\right)|u|^{q+1} \leq 0, \end{array}$$

and apply Theorem 1.

Example 2 *Let* Ω *be a bounded open of set* $\mathbb{R}^n$. *Then, problem*

$$\begin{cases} \frac{\partial}{\partial t}\left(e^{-t^2}\frac{\partial u}{\partial t}\right) - \Delta u = \theta(x)|u|^{p-1}u \text{ in } \mathbb{R}^+ \times \Omega, \\ \left(u + \varepsilon\frac{\partial u}{\partial n}\right)(t,\sigma) = 0 \text{ on } \mathbb{R}^+ \times \partial\Omega, \\ u(0,x) = 0, \forall x \in \Omega, \end{cases} \tag{4.2}$$

where

$$p \geq 1, \ \theta : \overline{\Omega} \to \mathbb{R}, \text{ is nonnegative,}$$

admits only the trivial solution, $u \equiv 0$.

Indeed,

$$\begin{array}{l} \lambda(t) = -e^{-t^2} < 0, \lambda'(t) = 2te^{-t^2} \geq 0, \forall t \geq 0, \\ 2F(x,u) - uf(x,u) = \theta(x)\left(\frac{2}{p+1} - 1\right)|u|^{p+1} \geq 0. \end{array}$$

Theorem 1 gives the result.

Example 3 *Let* Ω *be a bounded open of set* $\mathbb{R}^n$, $p, q \geq 1$,*Then, the system*

$$\begin{cases} -\frac{\partial}{\partial t}\left(\lambda(t)\frac{\partial u}{\partial t}\right) - \Delta u + (p+1)\theta(x)u|u|^{p-1}|v|^{q+1} = 0 \text{ in } \mathbb{R} \times \Omega, \\ -\frac{\partial}{\partial t}\left(\lambda(t)\frac{\partial v}{\partial t}\right) - \Delta v + (q+1)\theta(x)v|v|^{q-1}|u|^{p+1} = 0 \text{ in } \mathbb{R} \times \Omega, \\ \left(u + \varepsilon\frac{\partial u}{\partial n}\right)(t,\sigma) = \left(v + \varepsilon\frac{\partial v}{\partial n}\right)(t,\sigma) = 0 \text{ on } \mathbb{R} \times \partial\Omega, \end{cases} \tag{4.3}$$

where

$$\begin{array}{l} \theta : \overline{\Omega} \to \mathbb{R}, \text{is nonnegative,} \\ \lambda(t) > 0 \text{ of a class } L^\infty(\mathbb{R}), \end{array}$$

admit only the trivial solutions, $u \equiv v \equiv 0$.

Indeed, there exist a function F defined as follows

$$F(x,u,v)=\theta\left(x\right)\left|u\right|^{p+1}\left|v\right|^{q+1},$$

witch satisfies

$$\begin{aligned}
&\tfrac{\partial F}{\partial u}=f_1(x,u,v)=(p+1)\,\theta\left(x\right)u\left|u\right|^{p-1}\left|v\right|^{q+1},\\
&\tfrac{\partial F}{\partial v}=f_2(x,u,v)=(q+1)\,\theta\left(x\right)v\left|v\right|^{q-1}\left|u\right|^{p+1},\\
&F(x,u,v)-uf_1(x,u,v)-vf_2\left(x,u,v\right)=-\theta\left(x\right)\left(p+q+1\right)\left|u\right|^{p+1}\left|v\right|^{q+1}\leq 0.
\end{aligned}$$

Theorem 3 gives the result.

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